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Oct. 13	8 3	92·88	2	...	...	...	...	...	N
14	8 40	93·28	2	...	8 30	50·326	5	...	„
15	8 31	93·84	2	...	8 52	50·137	6	...	„
19	8 30	95·84	2	...	8 15	45·536	5	...	„
20	8 6	96·32	2	...	8 2	43·602	5	...	„
23	8 3	98·37	2	...	8 16	36·387	5	...	„
27	7 16	103·31	4	...	7 28	24·168	5	...	„

On an Investigation of Bessel's on the Refraction through an Atmosphere. By E. Neison.

In the *Astronomische Nachrichten*, No. 263, is an investigation of Bessel's on the limits of a possible lunar atmosphere, in which he determines the greatest possible density any atmosphere to the Moon can possess, and his conclusions have been widely accepted and quoted, though generally without any indications of how they were obtained, or where they appeared. Having lately become aware of the complete nature of Bessel's labours, their discordance with those given in the *Monthly Notices*, vol. xxxiii. p. 10, demanded attention. On examining Bessel's investigation, it was found that he had assumed the condition of a uniform temperature, as being the most favourable to a great surface density for a given horizontal refraction, whereas it is the most unfavourable; whilst it appeared certain he had, in quoting an equation from Laplace (*Méc. Cél.* iv. p. 252) employed one constituent without modifying it in the manner necessary when applied as he proceeded to do. As the uncertainty arising from such discrepancies is very inconvenient, an independent derivation of the true result has been made, omitting the constituent in question.

Assume

a = radius of Moon (or planet);

$g, \quad \delta, \quad p, \quad t, \quad \mu$ = force of gravity, density, pressure, temperature, and index of refraction, at a height x above the surface;

$g_o, \delta_o, p_o, t_o, \mu_o$ = the same, at the surface.

Then the equation

$$dp = -g.\delta.dx,$$

changing the variable to u, where $u = \dfrac{1}{a+x}$ and $dx = -\dfrac{du}{u^2}$, and putting for g its equivalent $a^2\,u^2\,g_o$, becomes

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$$dp = g_0 a^2 \delta \, du. \tag{1}$$

Dividing by the value of p from the equation

$$\frac{p}{p_o} = \frac{\delta}{\delta_o} \cdot \frac{1+\epsilon t}{1+\epsilon t_o}, \tag{2}$$

where ϵ is the rate of expansion of gases for one degree centigrade, the equation becomes

$$\frac{dp}{p} = g_o \frac{\delta_o}{p_o} a^2 \frac{1+\epsilon t_o}{1+\epsilon t} du; \tag{3}$$

and by integration and determining the constant from the condition that ϵ equals $\log p$ when $u = \frac{1}{a}$, we find

$$\frac{p}{p_o} = e^{g_o \frac{\delta_o}{p_o} (1+\epsilon t_o) \int \frac{du}{1+\epsilon t}}$$

where the inferior limit of the integral is $\frac{1}{a}$.

Hence

$$\frac{\delta}{\delta_o} = \frac{1+\epsilon t_o}{1+\epsilon t} e^{\frac{\delta_o}{p_o} (1+\epsilon t_o) \int \frac{du}{1+\epsilon t}}, \tag{4}$$

and then assuming the further condition that the temperature is constant, so that $t = t_o$, and restoring to u its value, we obtain

$$\frac{\delta}{\delta_o} = e^{-g_o \frac{\delta_o}{p_o} \cdot \frac{a}{a+x} \cdot x}; \tag{5}$$

or assuming $1-s = \frac{a}{a+x}$, and denoting the constant quantity $g_o \frac{\delta_o}{p_o}$ by β,

$$\delta = \delta_o e^{-\beta s}. \tag{6}$$

From the theory of the refraction of light through a spherical atmospheric envelope, if ζ be the refraction, i the angle of incidence, and z the zenith distance,

$$d\zeta = -\tan i \frac{d\mu}{\mu},$$

and

$$\mu (a+x) \sin i = \mu_o . a \sin z;$$

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whence, substituting in the former the value of tan i derived from the latter, and writing for $\frac{a}{a+x}$ the value $1-s$, after reduction the equation becomes

$$d\zeta = -\frac{(1-s)\sin z\,\frac{d\mu}{\mu}}{\left\{\cos^2 z - \left(1-\frac{\mu^2}{{\mu_o}^2}\right) + (2s-s^2)\sin^2 z\right\}^{\frac{1}{2}}}\,. \qquad (8)$$

From the law of the dependence of the refraction upon the density,

$$\mu^2 - 1 = \frac{2\alpha}{1-2\alpha}\cdot\frac{\delta}{\delta_o};$$

by differentiating, and (as α is supposed very small) replacing $2\alpha\left(1-\frac{\delta}{\delta_o}\right)$ by α,

$$\frac{d\mu}{\mu} = \frac{\alpha}{1-\alpha}\cdot\frac{d\delta}{\delta_o},$$

and then, by substituting in equation (8),

$$d\zeta = -\frac{\alpha}{1-\alpha}\cdot\frac{(1-s)\sin z\,\frac{d\delta}{\delta_o}}{\left\{\cos^2 z - 2\alpha\left(1-\frac{\delta}{\delta_o}\right) + (2s-s^2)\sin^2 z\right\}^{\frac{1}{2}}}, \qquad (9)$$

the differential equation to the refraction obtained by Laplace, (*Méc. Celeste*, Book x., c. 1., s. 5). As s is very small for any sensible refraction, its powers can be neglected, and the above becomes by development

$$d\zeta = -\frac{\alpha}{1-\alpha}\cdot\frac{\sin z\,\frac{d\delta}{\delta_o}}{\left\{\cos^2 z - 2\alpha\left(1-\frac{\delta}{\delta_o}\right) + 2s\sin^2 z\right\}^{\frac{1}{2}}} + s(A), \qquad (10)$$

where A is simply an abbreviation for the first term on the right, and is for the present disregarded. By differentiation and substitution from the law of decrease of density, the above becomes

$$d\zeta = \frac{\alpha}{1-\alpha}\cdot\frac{\beta\sin z\,.\,e^{-\beta s}\,ds}{\{\cos^2 z - 2\alpha\,(1-e^{-\beta s}) + 2s\sin^2 z)\}^{\frac{1}{2}}}:$$

the same equation as obtained by Laplace. Assume, for the purpose of integrating this equation, a new variable, y connected with s by the equation :

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$$s = y + \frac{\alpha\,(1-e^{-\beta s})}{\sin^2 z},$$

whence, by Lagrange's Theorem, expanding in powers of α

$$e^{-\beta s} = e^{-\beta y}\{1 - \text{terms involving } \alpha \text{ and its powers}\}. \tag{11}$$

As α is supposed exceedingly small, and on substitution any terms involving α would give rise to α^2, these can be neglected, whence on substitution,

$$d\zeta = \frac{\alpha}{1-\alpha} \cdot \frac{\beta \sin z\, e^{-\beta y}\, dy}{\{\cos^2 z + 2y \sin^2 z\}}\,.$$

By assuming $y = \frac{t^2}{\beta} - \frac{1}{2}\cot^2 z$ this reduces itself directly to a well-known form:

$$\zeta = \frac{\alpha}{1-\alpha}\sqrt{2\beta}\, e^{\frac{1}{2}\beta\cot^2 z}\int e^{-t^2}\, dt.$$

For the horizontal refraction $\cot^2 z$ becomes zero, and the limits of the integral are from 0 to ∞, the latter being allowable, as whatever magnitude may be assigned to s above a small value it cannot affect the refraction; and under these conditions the integral becomes $\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ or $\frac{\sqrt{\pi}}{2}$ and

$$\zeta = \frac{\alpha}{1-\alpha}\sqrt{\frac{\beta\pi}{2}}. \tag{12}$$

or Laplace's value neglecting powers of α. By retaining powers of α the above would have been the coefficient of a series, thus—

$$\{e^{-\alpha\beta} + 2^{\frac{1}{2}}\,\alpha\,.\,\beta\,\epsilon^{-2\alpha\beta} + \text{etc.}\}$$

By reducing the disregarded term in (10) in exactly the same manner, it becomes a similar transformation of the gamma function, and its value is $\frac{\alpha}{1-\alpha}\sqrt{\frac{\pi}{8\beta}}$ multiplied by the same series as the last, and thus it cannot be greater than $\frac{1}{2\beta}$ of the above term (12). On the Moon β is about 33.

Restoring in (12) the value of β, it becomes

$$\zeta = \frac{\alpha}{1-\alpha}\sqrt{\left\{g_0\frac{\delta_0}{p_0}\,.\,a\,.\,\frac{\pi}{2}\right\}}\,. \tag{13}$$

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The above value of α depends on the density of the air, and may be written $\alpha' \delta_0'$, where α' is the refraction of air for the standard density on the Earth, and δ_0' represents the density of the air in terms of the standard density at the Earth's surface where the horizontal refraction occurs. Disregarding powers of α, and transposing,

$$\delta_o' = \frac{\zeta}{\alpha'} \sqrt{\left\{\frac{2}{\pi} \cdot \frac{p_o}{g_o \cdot \delta_o \cdot a}\right\}} \cdot \tag{14}$$

Then, from the condition that the surface density shall be such that the density δ_0' at a height x above the surface shall not exceed a given value of ζ, by replacing the given value of δ_0' in equation (5),

$$\delta_o = \frac{\zeta}{\alpha'} \sqrt{\left\{\frac{2}{a\pi} \cdot \frac{p_o}{g_o \delta_o} \cdot \right\}}\, e^{g_o \frac{\delta_o}{p_o} \cdot \frac{a}{a+x} \cdot x} \tag{15}$$

and this is the maximum density at the surface for an atmosphere of uniform temperature whose horizontal refraction at the height x is equal to ζ.

The fundamental equation employed by Bessel, modifying slightly his notation, is,

$$\delta_o = \delta\, e^{\frac{m a_1^2}{l a^2} x},$$

obtainable from (5) by writing for $g_o \frac{\delta_o}{p_o}$ its value in terms of the gravity at the Earth's surface, by replacing $\frac{p_o}{\delta_o}$ by l and g_o by $m \frac{a_1^2}{a^2}$, where m is the mass of the Moon, and a the radius of the Earth, the constant quantity $\frac{p_o}{g_o \delta_o}$ being the height of the column of air of density δ_0 necessary under action of gravity g_0 to produce a pressure equal to p_0, whilst Bessel's l is the same when the force of gravity is supposed equal to that at the surface of the earth; and introducing the approximation employed by Bessel that $\frac{a\,x}{a+x}$ can be replaced by x. The laws of decrease of density found are therefore alike, so therefore should be the results. The expression for the refraction Bessel derives from the *Méc. Cel.* p. 252:

$$\zeta = \alpha \sqrt{\left\{\frac{a}{l} \cdot \frac{\pi}{2}\right\}} \cdot$$

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which is Laplace's result, obtained at (12) after writing $1 - \alpha$ as unity and replacing β by $\frac{a}{l}$, which is Laplace's expression; and then in the same way as (15) is obtained, Bessel finds

$$\delta_o = \frac{\zeta}{a'} \sqrt{\left\{\frac{2}{\pi} \cdot \frac{l}{a}\right\}} \, e^{\frac{m a_1^2}{l a^2} x}.$$

Here Bessel assumes that the value of l in Laplace's equation is identical with his own, but Laplace states $l = \frac{p_o}{g_o \delta_o}$; and to convert it into Bessel's constant l or $\frac{p_o}{\delta_o}$, it must be multiplied by g_o or $\frac{m a_1^2}{a^2}$, which though taken into account in obtaining the law of decrease of density Bessel has neglected to do in applying the equation for the refraction, and his result therefore is imperfect. Replacing Laplace's l, by its value in terms of Bessel's or $\frac{l a^2}{m a_1^2}$, the equation above becomes

$$\delta_o = \frac{\zeta}{a'} \sqrt{\left\{\frac{2}{a \pi} \cdot \frac{l a^2}{m a_1^2}\right\}} \, e^{\frac{m a_1^2}{l a^2} x},$$

which is identical with that obtained at (15) and converted into it by restoring to x its proper value $\frac{a \, x}{a + x}$ and replacing $\frac{l a^2}{m a^2}$ by its value $\frac{p_o}{g_o \delta_o}$, or Bessel's result must be increased by $\sqrt{\frac{1}{g_o}}$ or $\frac{a}{a_1 \sqrt{m}}$.

Bessel, employing Lindenau's value for the mass, and Burckhardt's for the diameter of the Moon, found the maximum surface-density for a lunar atmosphere exerting a horizontal refraction t the height of 5 miles above the surface of only one second of arc to be $\frac{1}{963}$rd of that of the Earth. Employing, however, the corrected expression, this is raised at once to $\frac{1}{377}$; but as already mentioned, the condition of a uniform temperature is not (as Bessel considered it to be) the most favourable for ensuring a moderately quick fall of temperature, and the surface density is raised at once to $\frac{1}{240}$th of that for the Earth, or more than four times as great as supposed possible by Bessel.

As in the paper by Bessel, the results of the foregoing are applicable to any planet as well as the Moon; but for a considerable

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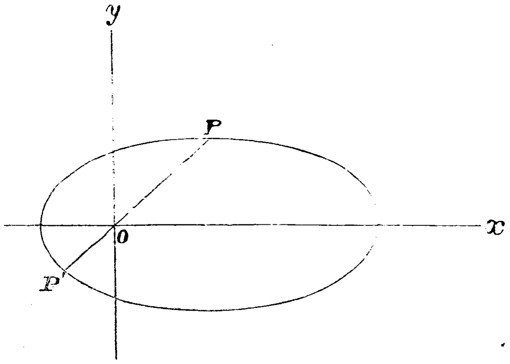
surface-density, if accuracy is required, it will be necessary to include the square of a, and for small planets the term s (A) in (10); in the larger planets this will be unnecessary, but the cube of a must be retained.

On a Theorem in Elliptic Motion. By Prof. Cayley.

Let a body move through apocentre between two opposite points of its orbit, say from the point P, excentric anomaly u, to the point P′, excentric anomaly u', where u, u' are each positive, $u < \pi$, $u' > \pi$. Taking the origin at the focus, and the axis of x in the direction through apocentre, then—



Coordinates of P are $x = a\,(-\cos u + e)$, $y = a\sqrt{1-e^2}\sin u$.

„ P′ „ $x = a\,(-\cos u' + e)$, $y = a\sqrt{1-e^2}\sin u'$;

whence, expressing that the points P, P′ are in a line with the focus,

$$\sin u'\,(-\cos u + e) - \sin u\,(-\cos u' + e) = 0,$$

that is

$$\sin\,(u'-u) = e\,(\sin u' - \sin u),$$

which is negative, viz. $u' - u$ is $> \pi$.

The time of passage from P to P′ is

$$\begin{aligned} n\,t &= (u' - e\sin u') - (u - e\sin u), \\ &= u' - u - e\,(\sin u' - \sin u), \\ &= u' - u - \sin\,(u' - u), \end{aligned}$$

which, $u' - u$ being greater than π, and $-\sin\,(u' - u)$ positive, is greater than π; viz. the time of passage is greater than one-half the periodic time. Of course if P, P′ are at pericentre and apocentre, the time of passage is equal one-half the periodic time.

The time of passage from P′ to P through pericentre is—

$$n\,\tau = 2\pi - (u' - u) + \sin\,(u' - u),$$

which is

$$= 2\pi - (u' - u) - \sin\,\{2\pi - (u' - u)\},$$